

THE GEOMETRY OF K-ORBITS OF A SUBCLASS OF MD5-GROUPS AND FOLIATIONS FORMED BY THEIR GENERIC K-ORBITS

Le Anh Vu* and Duong Minh Thanh**

** Department of Mathematics and Informatics

University of Pedagogy, Ho Chi Minh City, Vietnam

e-mail: leanhvu@hcmup.edu.vn

*** Department of Physics

University of Pedagogy, Ho Chi Minh City, Vietnam

e-mail: dmthanh@hcmup.edu.vn

Abstract

The present paper is a continuation of Le Anh Vu's ones [13], [14], [15]. Specifically, the paper is concerned with the subclass of

⁰**Key words:** Lie group, Lie algebra, MD5-group, MD5-algebra, K-orbits, foliation, transverse measure, measurable foliation.

2000AMS Mathematics Subject Classification: Primary 22E45, Secondary 46E25, 20C20.

connected and simply connected MD5-groups such that their MD5-algebras \mathcal{G} have the derived ideal $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}] \equiv \mathbf{R}^3$. We shall describe the geometry of K-orbits of these MD5-groups. The foliations formed by K-orbits of maximal dimension of these MD5-groups and their measurability are also presented in the paper.

Introduction

It is well known that the Kirillov's method of orbits (see [3]) plays the most important in the theory of representations of Lie groups. By this method, we can obtain all the unitary irreducible representations of solvable and simply connected Lie groups. The importance of Kirillov's method of orbits is the co-adjoint representation (K-representation). Therefore, it is meaningful to study the K-representation in the theory of representations of solvable Lie groups.

The structure of solvable Lie groups and Lie algebras is not to complicated, although the complete classification of them is unresolved up to now. In 1980, studying the Kirillov's method of orbits, D. N. Diep (see [2], [6]) introduced the class of Lie groups and Lie algebras MD. Let G be an n -dimensional Lie group. It is called an MD n -group (see [2], [6]), iff its orbits in the co-adjoint representation (K-orbits) are orbits of dimension zero or maximal dimension. The corresponding Lie algebra are called MD n -algebra. It is worth noticing that, for each connected MD n -group, the family of all K-orbits of maximal dimension forms a foliation. Therefore, we can combine the study of MD n -algebras and MD n -groups with the method of Connes (see [1]) in the theory of foliations and operator algebras.

All MD4-algebras were first listed by D.V Tra in 1984 (see [7]) and then classified up to an isomorphism by the first author in 1990 (see [10], [11]). The description of the geometry of K-orbits of all indecomposable MD4-groups,

the topological classification of foliations formed by K-orbits of maximal dimension and the characterization of C*-algebras associated to these foliations by the method of K-functors were also given by the first author in 1990 (see [8], [9], [10], [11]). In 2001, Nguyen Viet Hai (see [3]) introduced deformation quantization on K-orbits of MD4-groups and obtained all representations of MD4-groups. Until now, no complete classification of MDn-algebras with $n \geq 5$ is known.

Recently, the first author continued study MD5-groups and MD5-algebras \mathcal{G} in cases $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}] \equiv \mathbf{R}^k$; $k = 1, 2$. (see [13], [14]). In the present paper we concern with a similar problem for different MD5-groups and MD5-algebras \mathcal{G} in the case $\mathcal{G}^1 \equiv \mathbf{R}^3$. We begin our discussion in Section 2 by repeating this subclass of MD5-algebras which is listed in [15] by the first author. Section 3 is devoted to the geometric description of K-orbits of MD5-groups corresponding to these MD5-algebras and a discussion of the foliations formed by their maximal dimensional K-orbits. At first, we recall in Section 1 some preliminary results and notations which will be used later. For details we refer the reader to References [1], [2], [4].

1 Preliminaries

1.1 The co-adjoint Representation and K-orbits of a Lie Group

Let G be a Lie group. We denote by \mathcal{G} the Lie algebra of G and by \mathcal{G}^* the dual space of \mathcal{G} . To each element g of G we associate an automorphism

$$A_{(g)} : G \longrightarrow G$$

$$x \longmapsto A_{(g)}(x) := gxg^{-1}.$$

(which is called the internal automorphism associated to g). $A_{(g)}$ induces the tangent map

$$A_{(g)*} : \mathcal{G} \longrightarrow \mathcal{G}$$

$$X \longmapsto A_{(g)*}(X) := \left. \frac{d}{dt} [g \cdot \exp(tX) g^{-1}] \right|_{t=0} .$$

Definition 1.1.1. *The action*

$$Ad : G \longrightarrow Aut(\mathcal{G})$$

$$g \longmapsto Ad(g) := A_{(g)*}$$

is called the adjoint representation of G in \mathcal{G} .

Definition 1.1.2. *The action*

$$K : G \longrightarrow Aut(\mathcal{G}^*)$$

$$g \longmapsto K_{(g)}$$

such that

$$\langle K_{(g)}F, X \rangle := \langle F, Ad(g^{-1})X \rangle; \quad (F \in \mathcal{G}^*, X \in \mathcal{G})$$

is called the co-adjoint representation of G in \mathcal{G}^ .*

Definition 1.1.3. *Each orbit of the co-adjoint representation of G is called a K -orbit. The dimension of a K -orbit of G is always even.*

1.2 Foliations and Measurable Foliations

Let V be a smooth manifold. We denote by TV its tangent bundle, so that for each $x \in V$, $T_x V$ is the tangent space of V at x .

Definition 1.2.1. *A smooth subbundle \mathcal{F} of TV is called integrable iff the following condition is satisfied: every x from V is contained in a submanifold W of V such that $T_p(W) = \mathcal{F}_p$ ($\forall p \in W$).*

Definition 1.2.2. A foliation (V, \mathcal{F}) is given by a smooth manifold V and an integrable subbundle \mathcal{F} of TV . Then, V is called the foliated manifold and \mathcal{F} is called the subbundle defining the foliation.

Definition 1.2.3. The leaves of the foliation (V, \mathcal{F}) are the maximal connected submanifolds L of V with $T_x(L) = \mathcal{F}_x$ ($\forall x \in L$).

The set of leaves with the quotient topology is denoted by V/\mathcal{F} and called the *space of leaves* of (V, \mathcal{F}) . It is a fairly untractable topological space.

The partition of V in leaves $V = \bigcup_{\alpha \in V/\mathcal{F}} L_\alpha$ is characterized geometrically by the following local triviality: Every $x \in V$ has a system of local coordinates $\{U; x^1, x^2, \dots, x^n\} (x \in U; n = \dim \mathcal{F})$ so that for any leaf L with $L \cap U \neq \emptyset$, each connected component of $L \cap U$ (which is called a *plaque* of the leaf L) is given by the equations

$$x^{k+1} = c^1, x^{k+2} = c^2, \dots, x^n = c^{n-k}; k = \dim \mathcal{F}$$

where c^1, c^2, \dots, c^{n-k} are constants (depending on each plaque). Such a system $\{U, x^1, x^2, \dots, x^n\}$ is called a *foliation chart*.

A foliation can be given by a partition of V in a family \mathcal{C} of its submanifolds such that each $L \in \mathcal{C}$ is a maximal connected integral submanifold of some integrable subbundle \mathcal{F} of TV . Then \mathcal{C} is the family of leaves of the foliation (V, \mathcal{F}) . Sometimes \mathcal{C} is identified with \mathcal{F} and we will say that (V, \mathcal{F}) is formed by \mathcal{C} .

Definition 1.2.4. A submanifold N of the foliated manifold V is called a *transversal* iff $T_x V = T_x N \oplus \mathcal{F}_x$ ($\forall x \in N$). Thus, $\dim N = n - \dim \mathcal{F} = \text{codim} \mathcal{F}$.

A Borel subset B of V such that $B \cap L$ is countable for any leaf L is called a *Borel transversal* to (V, \mathcal{F}) .

Definition 1.2.5. A transverse measure Λ for the foliation (V, \mathcal{F}) is σ -additive map $B \mapsto \Lambda(B)$ from the set of all Borel transversals to $[0, +\infty]$ such that the following conditions are satisfied :

(i) If $\psi : B_1 \rightarrow B_2$ is a Borel bijection and $\psi(x)$ is on the leaf of any $x \in B_1$, then $\Lambda(B_1) = \Lambda(B_2)$.

(ii) $\Lambda(K) < +\infty$ if K is any compact subset of a smooth transversal submanifold of V .

By a measurable foliation we mean a foliation (V, \mathcal{F}) equipped with some transverse measure Λ .

Let (V, \mathcal{F}) be a foliation with \mathcal{F} is oriented. Then the complement of zero section of the bundle $\Lambda^k(\mathcal{F})$ ($k = \dim \mathcal{F}$) has two components $\Lambda^k(\mathcal{F})^-$ and $\Lambda^k(\mathcal{F})^+$.

Let μ be a measure on V and $\{U, x^1, x^2, \dots, x^n\}$ be a foliation chart of (V, \mathcal{F}) . Then U can be identified with the direct product $N \times \Pi$ of some smooth transversal submanifold N of V and a some plaque Π . The restriction of μ on $U \equiv N \times \Pi$ becomes the product $\mu_N \times \mu_\Pi$ of measures μ_N and μ_Π respectively.

Let $X \in C^\infty(\Lambda^k(\mathcal{F}))^+$ be a smooth k -vector field and μ_X be the measure on each leaf L determined by the volume element X .

Definition 1.2.6. The measure μ is called X -invariant iff μ_X is proportional to μ_Π for an arbitrary foliation chart $\{U, x^1, x^2, \dots, x^n\}$.

Let $(X, \mu), (Y, \nu)$ be two pairs where $X, Y \in C^\infty(\Lambda^k(\mathcal{F}))^+$ and μ, ν are measures on V such that μ is X -invariant, ν is Y -invariant.

Definition 1.2.7. $(X, \mu), (Y, \nu)$ are equivalent iff $Y = \varphi X$ and $\mu = \varphi \nu$ for some $\varphi \in C^\infty(V)$.

There is one bijective map between the set of transverse measures for (V, \mathcal{F}) and the one of equivalence classes of pairs (X, μ) , where $X \in C^\infty(\Lambda^k(\mathcal{F}))^+$ and μ is a X -invariant measure on V .

Thus, to prove that (V, \mathcal{F}) is measurable, we only need choose some suitable pair (X, μ) on V .

2 A Subclass of Indecomposable MD5-Algebras and MD5-Groups

From now on, G will denote a connected simply-connected solvable Lie group of dimension 5. The Lie algebra of G is denoted by \mathcal{G} . We always choose a fixed basis $(X_1, X_2, X_3, X_4, X_5)$ in \mathcal{G} . Then Lie algebra \mathcal{G} isomorphic to \mathbf{R}^5 as a real vector space. The notation \mathcal{G}^* will mean the dual space of \mathcal{G} . Clearly \mathcal{G}^* can be identified with \mathbf{R}^5 by fixing in it the basis $(X_1^*, X_2^*, X_3^*, X_4^*, X_5^*)$ dual to the basis $(X_1, X_2, X_3, X_4, X_5)$.

Recall that a group G is called a *MD5-group* iff its K -orbits are orbits of dimension zero or maximal dimension. Then its Lie algebra is called *MD5 - algebra*. Note that for any MD n - algebra \mathcal{G}_0 ($0 < n < 5$), the direct sum $\mathcal{G} = \mathcal{G}_0 \oplus \mathbf{R}^{5-n}$ of \mathcal{G}_0 and the commutative Lie algebra \mathbf{R}^{5-n} is a MD5-algebra. It is called a *decomposable* MD5 - algebra, the study of which can be directly reduced to the case of MD n - algebras with ($0 < n < 5$). Therefore, we will restrict on the case of *indecomposable* MD5 - algebras.

2.1 List of Indecomposable MD5 - Algebras and MD5 - Groups

For the sake of convenience, we shall continue using the first author's notations in [10], [11], [12], [15]. Specifically, we consider the set $\{\mathcal{G}_{5,3,1(\lambda_1, \lambda_2)}, \mathcal{G}_{5,3,2(\lambda)}, \mathcal{G}_{5,3,3(\lambda)}, \mathcal{G}_{5,3,4}, \mathcal{G}_{5,3,5(\lambda)}, \mathcal{G}_{5,3,6(\lambda)}, \mathcal{G}_{5,3,7}, \mathcal{G}_{5,3,8(\lambda, \varphi)}\}$ of solvable Lie algebras of dimension 5 which are listed by the first author in [15]. Each algebra

\mathcal{G} from this set has

$$\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}] = \mathbf{R}.X_3 \oplus \mathbf{R}.X_4 \oplus \mathbf{R}.X_5 \equiv \mathbf{R}^3; [X_1, X_2] = X_3; ad_{X_1} = 0.$$

The operator $ad_{X_2} \in End(\mathcal{G}^1) \equiv Mat(3, \mathbf{R})$ is given as follows:

1. $\mathcal{G}_{5,3,1(\lambda_1, \lambda_2)}$:

$$ad_{X_2} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda_1, \lambda_2 \in \mathbf{R} \setminus \{1\}, \lambda_1 \neq \lambda_2 \neq 0.$$

2. $\mathcal{G}_{5,3,2(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbf{R} \setminus \{0, 1\}.$$

3. $\mathcal{G}_{5,3,3(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbf{R} \setminus \{1\}.$$

4. $\mathcal{G}_{5,3,4}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. $\mathcal{G}_{5,3,5(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbf{R} \setminus \{1\}.$$

6. $\mathcal{G}_{5,3,6(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbf{R} \setminus \{0, 1\}.$$

7. $\mathcal{G}_{5,3,7}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

8. $\mathcal{G}_{5,3,8(\lambda,\varphi)}$:

$$ad_{X_2} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbf{R} \setminus \{0\}, \varphi \in (0, \pi).$$

So we obtain a set of connected and simply-connected solvable Lie groups corresponding to the set of Lie algebras listed above. For convenience, each such Lie group is also denoted by the same indices as its Lie algebra. For example, $G_{5,3,6(\lambda)}$ is the connected and simply-connected Lie group corresponding to $\mathcal{G}_{5,3,6(\lambda)}$.

2.2 Remarks

In [15], the first author proved that all of the Lie algebras and Lie groups as were listed above are indecomposable MD5-algebras and MD5-groups. However, this assertion can be verified by the pictures of K-orbits of considered Lie groups in the next section.

3 The Main Results

3.1 The Geometry of K-orbits of considered Lie groups

Throughout this section, G will denote one of the groups $G_{5,3,1(\lambda_1,\lambda_2)}$, $G_{5,3,2(\lambda)}$, $G_{5,3,3(\lambda)}$, $G_{5,3,4}$, $G_{5,3,5(\lambda)}$, $G_{5,3,6(\lambda)}$, $G_{5,3,7}$, $G_{5,3,8(\lambda,\varphi)}$, \mathcal{G} for its Lie algebra, $\mathcal{G} = \langle X_1, X_2, X_3, X_4, X_5 \rangle$. $\mathcal{G}^* = \langle X_1^*, X_2^*, X_3^*, X_4^*, X_5^* \rangle \equiv \mathbf{R}^5$ is the dual space of \mathcal{G} , $F = \alpha X_1^* + \beta X_2^* + \gamma X_3^* + \delta X_4^* + \sigma X_5^* \equiv (\alpha, \beta, \gamma, \delta, \sigma)$ an arbitrary element of \mathcal{G}^* , and finally Ω_F the K-orbit of G which contains F .

Proposition 3.1.1. *If $G = G_{5,3,1(\lambda_1,\lambda_2)}$ then the picture of the K-orbits of G be described as follows:*

1. *If $\gamma = \delta = \sigma = 0$ then*

$$\Omega_F = \{F(\alpha, \beta, 0, 0, 0)\}.$$

(K-orbit of dimension zero).

2. *If $\gamma = \delta = 0, \sigma \neq 0$ then*

$$\Omega_F = \{F(\alpha, y, 0, 0, s) : \sigma s > 0\}$$

(a part of 2-dimensional plane).

3. *If $\gamma = 0, \delta \neq 0, \sigma = 0$ then*

$$\Omega_F = \{F(\alpha, y, 0, t, 0) : \delta t > 0\}$$

(a part of 2-dimensional plane).

4. *If $\gamma = 0, \delta \neq 0, \sigma \neq 0$ then*

$$\Omega_F = \{F(\alpha, y, 0, t, s) : t = \delta \left(\frac{s}{\sigma}\right)^{\lambda_2}, \sigma s > 0\}$$

(a 2-dimensional cylinder).

5. If $\gamma \neq 0, \delta = \sigma = 0$ then

$$\Omega_F = \{F(x, y, z, 0, 0) : \lambda_1 x = \lambda_1 \alpha + \gamma - z, \gamma z > 0\}$$

(a part of 2-dimensional plane).

6. If $\gamma \neq 0, \delta = 0, \sigma \neq 0$ then

$$\Omega_F = \{F(x, y, z, 0, s) : \lambda_1 x = \lambda_1 \alpha + \gamma - z, \lambda_1 x = \lambda_1 \alpha + \gamma(1 - (\frac{s}{\sigma})^{\lambda_1}), \\ \sigma s > 0\}$$

(a 2-dimensional cylinder).

7. If $\gamma \neq 0, \delta \neq 0, \sigma = 0$ then

$$\Omega_F = \{F(x, y, z, t, 0) : \lambda_1 x = \lambda_1 \alpha + \gamma - z, \lambda_1 x = \lambda_1 \alpha + \gamma(1 - (\frac{t}{\delta})^{\frac{\lambda_1}{\lambda_2}}), \\ \delta t > 0\}$$

(a 2-dimensional cylinder).

8. If $\gamma \neq 0, \delta \neq 0, \sigma \neq 0$ then

$$\Omega_F = \{F(x, y, z, t, s) : \lambda_1 x = \lambda_1 \alpha + \gamma - z, \lambda_1 x = \lambda_1 \alpha + \gamma(1 - (\frac{s}{\sigma})^{\lambda_1}), \\ t = \delta(\frac{s}{\sigma})^{\lambda_2}, \sigma s > 0\}$$

(a 2-dimensional cylinder).

Sketch of the proof of Propositions 3.1.1

For G , we denote the set $\{F_U \in \mathcal{G}^*/U \in \mathcal{G}\}$ by $\Omega_F(\mathcal{G})$, where F_U is the linear form on the Lie algebra \mathcal{G} of G defined by

$$\langle F_U, A \rangle = \langle F, \exp(ad_U)(A) \rangle, A, U \in \mathcal{G}.$$

Let $U = a.X_1 + b.X_2 + c.X_3 + d.X_4 + f.X_5$ be an arbitrary of \mathcal{G} ; where $a, b, c, d, f \in \mathbf{R}$. Upon direct computation, we get :

$$\exp(ad_U) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\sum_{n=1}^{\infty} \frac{b^n \lambda_1^{n-1}}{n!} & (a - c\lambda_1) \sum_{n=1}^{\infty} \frac{(b\lambda_1)^{n-1}}{n!} & e^{b\lambda_1} & 0 & 0 \\ 0 & -d \sum_{n=1}^{\infty} \frac{b^{n-1} \lambda_1^n}{n!} & 0 & e^{b\lambda_2} & 0 \\ 0 & -f \sum_{n=1}^{\infty} \frac{b^{n-1}}{n!} & 0 & 0 & e^b \end{pmatrix}.$$

Thus, F_U is given as follows:

$$x = \alpha - \gamma \sum_{n=1}^{\infty} \frac{b^n \lambda_1^{n-1}}{n!};$$

$$y = \beta + \gamma(a - c\lambda_1) \sum_{n=1}^{\infty} \frac{(b\lambda_1)^{n-1}}{n!} - \delta d \sum_{n=1}^{\infty} \frac{b^{n-1} \lambda_1^n}{n!} - \sigma f \sum_{n=1}^{\infty} \frac{b^{n-1}}{n!};$$

$$z = \gamma e^{b\lambda_1};$$

$$t = \delta e^{b\lambda_2};$$

$$s = \sigma e^b.$$

So $\Omega_F(\mathcal{G})$ is described and the equation $\Omega_F(\mathcal{G}) = \Omega_F$ is verified by the same method presented in [8], [10], and [12]. \square

According to the method of the proof of Proposition 1, we get the following results.

Proposition 3.1.2. *If $G = G_{5,3,2(\lambda)}$ then the picture of the K -orbits of G be described as follows:*

1. *If $\gamma = \delta = \sigma = 0$ then*

$$\Omega_F = \{F(\alpha, \beta, 0, 0, 0)\}.$$

(K -orbit of dimension zero).

2. If $\gamma = \delta = 0, \sigma \neq 0$ then

$$\Omega_F = \{F(\alpha, y, 0, 0, s) : \sigma s > 0\}$$

(a part of 2-dimensional plane).

3. If $\gamma = 0, \delta \neq 0, \sigma = 0$ then

$$\Omega_F = \{F(\alpha, y, 0, t, 0) : \delta t > 0\}$$

(a part of 2-dimensional plane).

4. If $\gamma = 0, \delta \neq 0, \sigma \neq 0$ then

$$\Omega_F = \{F(\alpha, y, 0, t, s) : s = \sigma(\frac{t}{\delta})^\lambda, \delta t > 0\}$$

(a 2-dimensional cylinder).

5. If $\gamma \neq 0, \delta = \sigma = 0$ then

$$\Omega_F = \{F(x, y, z, 0, 0) : x = \alpha + \gamma - z, \gamma z > 0\}$$

(a part of 2-dimensional plane).

6. If $\gamma \neq 0, \delta = 0, \sigma \neq 0$ then

$$\Omega_F = \{F(x, y, z, 0, s) : x = \alpha + \gamma - z, s = \sigma(\frac{z}{\gamma})^\lambda, \gamma z > 0\}$$

(a 2-dimensional cylinder).

7. If $\gamma \neq 0, \delta \neq 0, \sigma = 0$ then

$$\Omega_F = \{F(x, y, z, t, 0) : x = \alpha + \gamma - z, x = \alpha + (1 - \frac{t}{\delta})\gamma, \delta t > 0\}$$

(a part of 2-dimensional plane).

8. If $\gamma \neq 0, \delta \neq 0, \sigma \neq 0$ then

$$\Omega_F = \{F(x, y, z, t, s) : x = \alpha + \gamma - z, x = \alpha + (1 - \frac{t}{\delta})\gamma, s = \sigma(\frac{t}{\delta})^\lambda, \delta t > 0\}$$

(a 2-dimensional cylinder). □

Proposition 3.1.3. *If $G = G_{5,3,3(\lambda)}$ then the picture of the K -orbits of G be described as follows:*

1. *If $\gamma = \delta = \sigma = 0$ then*

$$\Omega_F = \{F(\alpha, \beta, 0, 0, 0)\}.$$

(K -orbit of dimension zero).

2. *If $\gamma = \delta = 0, \sigma \neq 0$ then*

$$\Omega_F = \{F(\alpha, y, 0, 0, s) : \sigma s > 0\}$$

(a part of 2-dimensional plane).

3. *If $\gamma = 0, \delta \neq 0, \sigma = 0$ then*

$$\Omega_F = \{F(\alpha, y, 0, t, 0) : \delta t > 0\}$$

(a part of 2-dimensional plane).

4. *If $\gamma = 0, \delta \neq 0, \sigma \neq 0$ then*

$$\Omega_F = \{F(\alpha, y, 0, t, s) : \delta s = \sigma t, \delta t > 0\}$$

(a 2-dimensional cylinder).

5. *If $\gamma \neq 0, \delta = \sigma = 0$ then*

$$\Omega_F = \{F(x, y, z, 0, 0) : \lambda x = \lambda\alpha + \gamma - z, \gamma z > 0\}$$

(a part of 2-dimensional plane).

6. *If $\gamma \neq 0, \delta = 0, \sigma \neq 0$ then*

$$\Omega_F = \{F(x, y, z, 0, s) : \lambda x = \lambda\alpha + \gamma - z, \lambda x = \lambda\alpha + \gamma(1 - (\frac{s}{\sigma})^\lambda), \sigma s > 0\}$$

(a 2-dimensional cylinder).

7. If $\gamma \neq 0, \delta \neq 0, \sigma = 0$ then

$$\Omega_F = \{F(x, y, z, t, 0) : \lambda x = \lambda\alpha + \gamma - z, z = \gamma(\frac{t}{\delta})^\lambda, \delta t > 0\}$$

(a 2-dimensional cylinder).

8. If $\gamma \neq 0, \delta \neq 0, \sigma \neq 0$ then

$$\Omega_F = \{F(x, y, z, t, s) : \lambda x = \lambda\alpha + \gamma - z, \lambda x = \lambda\alpha + \gamma(1 - (\frac{t}{\delta})^\lambda), \\ \sigma t = \delta s, \delta t > 0\}$$

(a 2-dimensional cylinder).

□

Proposition 3.1.4. If $G = G_{5,3,4}$ then the picture of the K -orbits of G be described as follows:

1. If $\gamma = \delta = \sigma = 0$ then

$$\Omega_F = \{F(\alpha, \beta, 0, 0, 0)\}.$$

(K -orbit of dimension zero).

2. If $\gamma = \delta = 0, \sigma \neq 0$ then

$$\Omega_F = \{F(\alpha, y, 0, 0, s) : \sigma s > 0\}$$

(a part of 2-dimensional plane).

3. If $\gamma = 0, \delta \neq 0, \sigma = 0$ then

$$\Omega_F = \{F(\alpha, y, 0, t, 0) : \delta t > 0\}$$

(a part of 2-dimensional plane).

4. If $\gamma = 0, \delta \neq 0, \sigma \neq 0$ then

$$\Omega_F = \{F(\alpha, y, 0, t, s) : \delta s = \sigma t, \delta t > 0\}$$

(a part of 2-dimensional plane).

5. If $\gamma \neq 0, \delta = \sigma = 0$ then

$$\Omega_F = \{F(x, y, z, 0, 0) : x = \alpha + \gamma - z, \gamma z > 0\}$$

(a part of 2-dimensional plane).

6. If $\gamma \neq 0, \delta = 0, \sigma \neq 0$ then

$$\Omega_F = \{F(x, y, z, 0, s) : x = \alpha + \gamma - z, x = \alpha + \gamma(1 - \frac{s}{\sigma}), \sigma s > 0\}$$

(a part of 2-dimensional plane).

7. If $\gamma \neq 0, \delta \neq 0, \sigma = 0$ then

$$\Omega_F = \{F(x, y, z, t, 0) : x = \alpha + \gamma - z, z = \gamma \frac{t}{\delta}, \delta t > 0\}$$

(a part of 2-dimensional plane).

8. If $\gamma \neq 0, \delta \neq 0, \sigma \neq 0$ then

$$\Omega_F = \{F(x, y, z, t, s) : x = \alpha + \gamma - z, x = \alpha + \gamma(1 - \frac{s}{\sigma}), \sigma t = \delta s, \delta t > 0\}$$

(a part of 2-dimensional plane). □

Proposition 3.1.5. *If $G = G_{5,3,5(\lambda)}$ then the picture of the K -orbits of G be described as follows:*

1. If $\gamma = \delta = \sigma = 0$ then

$$\Omega_F = \{F(\alpha, \beta, 0, 0, 0)\}.$$

(K -orbit of dimension zero).

2. If $\gamma = \delta = 0, \sigma \neq 0$ then

$$\Omega_F = \{F(\alpha, y, 0, 0, s) : \sigma s > 0\}$$

(a part of 2-dimensional plane).

3. If $\gamma = 0, \delta \neq 0, \sigma = 0$ then

$$\Omega_F = \{F(\alpha, y, 0, t, s) : s = t \ln \frac{t}{\delta}, \delta t > 0\}$$

(a 2-dimensional cylinder).

4. If $\gamma = 0, \delta \neq 0, \sigma \neq 0$ then

$$\Omega_F = \{F(\alpha, y, 0, t, s) : s = \sigma \frac{t}{\delta} + t \ln \frac{t}{\delta}, \delta t > 0\}$$

(a 2-dimensional cylinder).

5. If $\gamma \neq 0, \delta = \sigma = 0$ then

$$\Omega_F = \{F(x, y, z, 0, 0) : \lambda x = \lambda \alpha + \gamma - z, \gamma z > 0\}$$

(a part of 2-dimensional plane).

6. If $\gamma \neq 0, \delta = 0, \sigma \neq 0$ then

$$\Omega_F = \{F(x, y, z, 0, s) : \lambda x = \lambda \alpha + \gamma - z, \lambda x = \lambda \alpha + \gamma(1 - (\frac{s}{\sigma})^\lambda), \sigma s > 0\}$$

(a 2-dimensional cylinder).

7. If $\gamma \neq 0, \delta \neq 0, \sigma = 0$ then

$$\Omega_F = \{F(x, y, z, t, s) : \lambda x = \lambda \alpha + \gamma - z, z = \gamma(\frac{t}{\delta})^\lambda, s = t \ln \frac{t}{\delta}, \delta t > 0\}$$

(a 2-dimensional cylinder).

8. If $\gamma \neq 0, \delta \neq 0, \sigma \neq 0$ then

$$\begin{aligned} \Omega_F = \{F(x, y, z, t, s) : \lambda x = \lambda \alpha + \gamma - z, \lambda y = \lambda \alpha + \gamma(1 - (\frac{t}{\delta})^\lambda), \\ s = \sigma \frac{t}{\delta} + t \ln \frac{t}{\delta}, \delta t > 0\} \end{aligned}$$

(a 2-dimensional cylinder).

□

Proposition 3.1.6. *If $G = G_{5,3,6(\lambda)}$ then the picture of the K -orbits of G be described as follows:*

1. *If $\gamma = \delta = \sigma = 0$ then*

$$\Omega_F = \{F(\alpha, \beta, 0, 0, 0)\}.$$

(K -orbit of dimension zero).

2. *If $\gamma = \delta = 0, \sigma \neq 0$ then*

$$\Omega_F = \{F(\alpha, y, 0, 0, s) : \sigma s > 0\}$$

(a part of 2-dimensional plane).

3. *If $\gamma = 0, \delta \neq 0, \sigma = 0$ then*

$$\Omega_F = \{F(\alpha, y, 0, t, 0) : \delta t > 0\}$$

(a part of 2-dimensional plane).

4. *If $\gamma = 0, \delta \neq 0, \sigma \neq 0$ then*

$$\Omega_F = \{F(\alpha, y, 0, t, s) : s = \sigma\left(\frac{t}{\delta}\right)^\lambda, \delta t > 0\}$$

(a 2-dimensional cylinder).

5. *If $\gamma \neq 0, \delta = \sigma = 0$ then*

$$\Omega_F = \{F(x, y, z, t, 0) : x = \alpha + \gamma - z, t = z \ln \frac{z}{\gamma}, \gamma z > 0\}$$

(a 2-dimensional cylinder).

6. *If $\gamma \neq 0, \delta = 0, \sigma \neq 0$ then*

$$\Omega_F = \{F(x, y, z, t, s) : x = \alpha + \gamma - z, t = z \ln \frac{z}{\gamma}, s = \sigma\left(\frac{z}{\gamma}\right)^\lambda, \sigma s > 0\}$$

(a 2-dimensional cylinder).

7. If $\gamma \neq 0, \delta \neq 0, \sigma = 0$ then

$$\Omega_F = \{F(x, y, z, t, 0) : x = \alpha + \gamma - z, t = z \ln \frac{z}{\gamma} + \delta \frac{z}{\gamma}, \gamma z > 0\}$$

(a 2-dimensional cylinder).

8. If $\gamma \neq 0, \delta \neq 0, \sigma \neq 0$ then

$$\Omega_F = \{F(x, y, z, t, s) : x = \alpha + \gamma - z, t = z \ln \frac{z}{\gamma} + \delta \frac{z}{\gamma}, s = \sigma \left(\frac{z}{\gamma}\right)^\lambda, \gamma z > 0\}$$

(a 2-dimensional cylinder). \square

Proposition 3.1.7. If $G = G_{(5,3,7)}$ then the picture of the K -orbits of G be described as follows:

1. If $\gamma = \delta = \sigma = 0$ then

$$\Omega_F = \{F(\alpha, \beta, 0, 0, 0)\}.$$

(K -orbit of dimension zero).

2. If $\gamma = \delta = 0, \sigma \neq 0$ then

$$\Omega_F = \{F(\alpha, y, 0, 0, s) : \sigma s > 0\}$$

(a part of 2-dimensional plane).

3. If $\gamma = 0, \delta \neq 0, \sigma = 0$ then

$$\Omega_F = \{F(\alpha, y, 0, t, s) : s = t \ln \frac{t}{\delta}, \delta t > 0\}$$

(a 2-dimensional cylinder).

4. If $\gamma = 0, \delta \neq 0, \sigma \neq 0$ then

$$\Omega_F = \{F(\alpha, y, 0, t, s) : s = t \ln \frac{t}{\delta} + \sigma \frac{t}{\delta}, \delta t > 0\}$$

(a 2-dimensional cylinder).

5. If $\gamma \neq 0, \delta = \sigma = 0$ then

$$\Omega_F = \{F(x, y, z, t, s) : x = \alpha + \gamma - z, t = z \ln \frac{z}{\gamma}, s = \frac{z}{2} \ln^2 \frac{z}{\gamma}, \gamma z > 0\}$$

(a 2-dimensional cylinder).

6. If $\gamma \neq 0, \delta = 0, \sigma \neq 0$ then

$$\Omega_F = \{F(x, y, z, t, s) : x = \alpha + \gamma - z, t = z \ln \frac{z}{\gamma}, s = \frac{z}{2} \ln^2 \frac{z}{\gamma} + \sigma \frac{z}{\gamma}, \gamma z > 0\}$$

(a 2-dimensional cylinder).

7. If $\gamma \neq 0, \delta \neq 0, \sigma = 0$ then

$$\begin{aligned} \Omega_F = \{F(x, y, z, t, s) : x = \alpha + \gamma - z, t = z \ln \frac{z}{\gamma} + \delta \frac{z}{\gamma}, \\ s = \frac{z}{2} \ln^2 \frac{z}{\gamma} + \frac{\delta}{\gamma} z \ln \frac{z}{\gamma}, \gamma z > 0\} \end{aligned}$$

(a 2-dimensional cylinder).

8. If $\gamma \neq 0, \delta \neq 0, \sigma \neq 0$ then

$$\begin{aligned} \Omega_F = \{F(x, y, z, t, s) : x = \alpha + \gamma - z, t = z \ln \frac{z}{\gamma} + \delta \frac{z}{\gamma}, \\ s = \frac{z}{2} \ln^2 \frac{z}{\gamma} + \frac{\delta}{\gamma} z \ln \frac{z}{\gamma} + \sigma \frac{z}{\gamma}, \gamma z > 0\} \end{aligned}$$

(a 2-dimensional cylinder).

□

Proposition 3.1.8. If $G = G_{(5,3,8(\lambda,\varphi))}$ then the picture of the K -orbits of G be described as follows:

1. If $\gamma = \delta = \sigma = 0$ then

$$\Omega_F = \{F(\alpha, \beta, 0, 0, 0)\}.$$

(K -orbit of dimension zero).

2. If $\gamma = \delta = 0, \sigma \neq 0$ then

$$\Omega_F = \{F(\alpha, y, 0, 0, s) : \sigma s > 0\}$$

(a part of 2-dimensional plane).

3. If $\gamma^2 + \delta^2 \neq 0$ then

$$\Omega_F = \{F(x, y, z + it, s) = F(x, y, \gamma e^{be^{-i\varphi}} + \delta e^{be^{i\varphi}}, \sigma e^{\lambda b})\}$$

(a 2-dimensional cylinder). □

3.2 Remark and Corollary

Note that if G is not $G_{5,3,8(\lambda,\varphi)}$ then G is exponential (see [5]). Hence $\Omega_F = \Omega_F(\mathcal{G})$.

For $G_{5,3,8(\lambda,\varphi)}$, the equation $\Omega_F = \Omega_F(\mathcal{G})$ is verified by using [12, Lemma II.1.5].

As an immediate consequence of above propositions we have the following corollary.

Corollary 3.2.1. *All of $G_{5,3,1(\lambda_1,\lambda_2)}$, $G_{5,3,2(\lambda)}$, $G_{5,3,3(\lambda)}$, $G_{5,3,4}$, $G_{5,3,5(\lambda)}$, $G_{5,3,6(\lambda)}$, $G_{5,3,7}$, $G_{5,3,8(\lambda,\varphi)}$ are MD5-groups.* □

3.3 MD5-Foliations Associated to Considered MD5-Groups

Theorem 3.3.1. *Let $G \in \{ G_{5,3,1(\lambda_1,\lambda_2)}, G_{5,3,2(\lambda)}, G_{5,3,3(\lambda)}, G_{5,3,4}, G_{5,3,5(\lambda)}, G_{5,3,6(\lambda)}, G_{5,3,7}, G_{5,3,8(\lambda,\varphi)} \}$, \mathcal{F}_G be the family of all its K -orbits of maximal dimension and $V_G = \bigcup \{ \Omega / \Omega \in \mathcal{F}_G \}$. Then (V_G, \mathcal{F}_G) is a measurable foliation in the sense of Connes. We call it MD5-foliation associated to G .*

Sketch of the Proof of Theorem 3.3.1

The proof is analogous to the case of MD4-groups in [8], [10], [12] or the first examples of MD5-group in [13], [14]. First, we need to define a smooth tangent 2-vector field on the manifold V_G such that each K-orbit Ω from \mathcal{F}_G is a maximal connected integrable submanifold corresponding to it. As the next step, we have to show that the Lebegues measure is invariant for that 2-vector field. This steps can be complete by direct computations. We therefore omit detail comutations of the proof. \square

3.4 Concluding Remark

We conclude this paper with the following comments.

1. It should be note that the results of Propositions 3.1.1 - 3.1.8 and Theorem 3.3.1 still hold for all indecomposable connected (no-simply connected) MD5-groups corresponding to above-mentioned MD5-algebras.
2. There are some interesting questions can be raised for further study. That are giving the topological classification of considered MD5-foliations, characterizing C^* -algebras associated to these foliations, constructing deformation quantization on K-orbits of considered MD5-groups to obtain all representations of them, ... The authors will announce the new results of this questions in the next papers.

Acknowledgement

The authors would like take this opportunity to thank their teacher, Prof. DSc. Do Ngoc Diep for his excellent advice and support. Thanks are due also to the first author's colleague, Prof. Nguyen Van Sanh for his encouragement.

References

- [1] **A. Connes**, *A Survey of Foliations and Operator Algebras*, Proc. Symp. Pure Math., 38(1982), 521 - 628, Part I.
- [2] **Do Ngoc Diep**, *Method of Noncommutative Geometry for Group C^* -algebras*, Chapman and Hall / CRC Press Reseach Notes in Mathematics Series, #416, 1999.
- [3] **Nguyen Viet Hai**, *Quantum co-adjoint orbits of MD4-groups*, Vietnam J. Math., Vol. **29**, IS. 02/2001, pp. 131-158.
- [4] **A. A. Kirillov**, *Elements of the Theory of Representations*, Springer - Verlag, Berlin - Heidenberg - New York, 1976.
- [5] **M. Saito**, *Sur Certains Groupes de Lie Resolubles*, Sci. Papers of the College of General Education, Uni. of Tokyo, 7(1957), 1 -11, 157 - 168.
- [6] **V. M. Son et H. H. Viet**, *Sur la Structure des C^* -algebres d' une Classe de Groupes de Lie*, J. Operator Theory, 11(1984), 77 - 90.
- [7] **D. V. Tra**, *On the Lie Algebras of low dimension* , Sci. Pape6s of the 12th College of Institute of Math. Vietnam, Hanoi 1984 (in Vietnamese).
- [8] **Le Anh Vu**, *The Foliation Formed by the K-orbits of Maximal Dimension of the Real Diamond Group*, Vietnam J. Math., XV(1987), N^o 3, 7 - 10 (in Vietnamese).
- [9] **Le Anh Vu**, *On the Structure of the C^* -algebra of the Foliation Formed by the K-orbits of Maximal Dimension of the Real Diamond Group*, J. Operator Theory, 24(1990), 227 - 238.
- [10] **Le Anh Vu**, *On the Foliations Formed by the Generic K-orbits of the MD4-Groups*, Acta Math. Vietnam, N^o 2(1990), 39 - 55.

- [11] **Le Anh Vu**, *Foliations Formed by Orbits of Maximal Dimension in the Co-adjoint Representation of a Class of Solvable Lie Groups*, Vest. Moscow Uni., Math. Bulletin, Vol. 48(1993), N^o 3, 24 - 27.
- [12] **Le Anh Vu**, *Foliations by K-orbits of Maximal Dimension of the class of Lie Groups MD4*, Ph.D. Dissertation, Institute of Math. Vietnam (1990) (in Vietnamese).
- [13] **Le Anh Vu**, *Foliations Formed by Orbits of Maximal Dimension of Some MD5-Groups*, East-West J. of Mathematics, Vol.5, N^o 2 (2003), 159 - 168.
- [14] **Le Anh Vu and Nguyen Cong Tri**, *Some Examples on MD5 algebras and MD5 foliations Associated to Corresponding MD5 groups*, (to appear in Scientific Journal of University of Pedagogy of Ho Chi Minh city).
- [15] **Le Anh Vu**, *On a Subclass of 5-dimensional Solvable Lie Algebras Which Have 3-Dimensional Commutative Derived Ideal*, Communication in ICMA-MU, Bangkok, Thailand, December 15-17, 2005, (to appear).